‡ Residue.

Residue is used to compute integral.

If a is analytic. then

$$\int_\gamma f(z)\,dz = 0 \quad \text{if } \gamma \subseteq N_\varepsilon(a)$$

If a is an isolated singularity, then

$\int_\gamma f(z)\,dz$ usually not equals to 0 if $\gamma \subseteq N_\varepsilon(a)$

1. Residue.

Def:

$$\operatorname*{Res}_{z=a} f(z) = \frac{1}{2\pi i}\int_\gamma f(z)\,dz \qquad \gamma: |z-a| < \rho,\ 0<\rho<R$$

Recall that, $\operatorname*{Res}_{z=a} f(z)$ is a number having nothing to do w/ ρ.

and according to Laurent

$$\operatorname*{Res}_{z=a} f(z) = \frac{1}{2\pi i}\int_\gamma f(z)\,dz = a_{-1}$$

$\Rightarrow$ If a function has finite removable singularities. the residues are zero there.

2. Cauchy Residue Theorem

If in D, there are singularties $a_1, a_2, \cdots a_n$, but analytic elsewhere, $C = \partial D$.

then

$$\int_C f(z)\,dz = 2\pi i \sum_{i=1}^{n} \operatorname*{Res}_{z=a_i} f(z)$$

3. Compute Residue.

① Laurent expansion

$\because \operatorname*{Res}_{z=a} f(z) = a_{-1}$

We can use Laurent expansion and find out the coefficient for $\frac{1}{z-a}$

② Direct Method

Theorem

if a is $f(z)$'s n-order pole, and we can rewrite $f(z)$ as

$$f(z) = \frac{\varphi(z)}{(z-a)^n} \qquad \varphi(z) \text{ is analytic at } a,\ \varphi(a) \neq 0.$$

then

$$\operatorname*{Res}_{z=a} f(z) = \frac{\varphi^{(n-1)}(a)}{(n-1)!}$$

Proof: $\operatorname*{Res}_{z=a} f(z) = \frac{1}{2\pi i}\int_\gamma \frac{\varphi(z)}{(z-a)^n}\,dz = \frac{\varphi^{(n-1)}(a)}{(n-1)!}$ topic 4, 11-①

Corollary a is $f(z)$'s 1-st order singularity

$\varphi(z) = (z-a) f(z)$

Then $\operatorname*{Res}_{z=a} f(z) = \varphi(a)$

Corollary a is $f(z)$'s 2-nd order singularity

$\varphi(z) = (z-a)^2 f(z)$

Then, $\operatorname*{Res}_{z=a} f(z) = \varphi'(a)$

Ex. $\int_{|z|=2} \frac{5z-2}{z(z-1)^2} dz$

$z=0$, $z=1$ are two singularities

$\operatorname*{Res}_{z=0} f(z) = \left.\frac{5z-2}{(z-1)^2}\right|_{z=0} = -2.$

$\operatorname*{Res}_{z=1} f(z) = \left.\left(\frac{5z-2}{z}\right)'\right|_{z=1} = \left.\frac{2}{z^2}\right|_{z=1} = 2$

$\therefore \int_{|z|=2} \frac{5z-2}{z(z-1)^2} dz = 2\pi i \cdot (-2+2) = 0$